Newton-Euler First-Moment Gravity Compensation

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1 Introduction

This document briefly describes a method for gravity compensation applicable to any rigid-body robotic manipulator in an open kinematic chain (i.e. only one connection to ground).

This method is designed to operate on the Barrett WAM™, an advanced torque-controlled robot.

The method consists of two steps. First, in the **calibration** step, the robot is made to hold a number of distinct poses, while torque measurements at each of the joints are taken. From these torque measurements, a vector μ is determined for each link j, using a linear regression. This vector μ_j, the *cumulative first moment of the mass*, has units of $\{\text{mass} \cdot \text{length}\}$, and represents the sum of link j's mass moment and the mass moments of all subsequent links. Expressed in joint coordinates, the vectors μ_j are pose-independent.

Once the μ vectors for the robot's mass configuration are determined, it is simple to derive the necessary torques for each joint, starting from the last link and recursively moving to the first one. Thus, in the **compensation** scheme, torques can be provided to each joint to gravity-compensate the robot, relative to whatever mass configuration was calibrated.

For this step, it is assumed that the robot is capable of maintaining a fixed position; for this purpose, the WAM™uses a set of simple PID controllers in joint-space.

2 Newton-Euler Formulation

2.1 Conventions

We will use the following conventions in this document, which are based on the Denavit-Hartenberg conventions.

The robot is made up of n moving rigid links, numbered $j = 1, 2, ...n$. Rigidly attached to the end of each link is an origin frame, where the z axis of frame j is the axis of rotation for the joint between link j and link $j+1$. The world frame is labeled frame 0, and for now, it is assumed that frame 0 is an inertial frame.

m_j	=	the mass of link j
r_{mj}	=	the location of the joint's point of rotation
r_{cj}	=	the location of the center of mass
g^j	=	the gravity vector in frame j
f_j	=	the force from link $j-1$ to link j
f_{j+1}	=	the force from link j to link $j+1$
$\vec{\tau}_j$	=	the torque from link $j-1$ to link j
$\vec{\tau}_{j+1}$	=	the torque from link j to link $j+1$
a_{cj}	=	the COM acceleration in frame j
α_j	=	the COM ang acceleration in frame j
ω_j	=	the COM ang velocity in frame j
F^j	=	an arbitrary external force in frame j
T^j	=	an arbitrary external torque in frame j

2.2 Generic Equations

From Spong, pp 276-277, we have the following equations of motion for a rigid link:

$$f_j - \mathrm{R}^j_{j+1} f_{j+1} + m_j g^j + F^j = m_j a^j_{cj} \tag{1}$$

$$\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + f_j \times (r_{cj} - r_{mj}) - \left(\mathrm{R}^j_{j+1} f_{j+1}\right) \times r_{cj} + T^j = I_j \alpha_j + \omega_j \times (I_j \omega_j) \tag{2}$$

Note that a number of terms have been adjusted from Spong to fit into the Denavit-Hartenberg conventions.

2.3 The Static Case

Next, we make the assumption that (a) there are no external forces or torques (F and T are zero) and (b) the system is static (a, α, and ω are zero). This leaves us with:

$$f_j = -m_j g^j + \mathrm{R}^j_{j+1} f_{j+1} \tag{3}$$

$$\vec{\tau}_j - \mathrm{R}^j_{j+1}\tau_{j+1} = -f_j \times (r_{cj} - r_{mj}) + \left(\mathrm{R}^j_{j+1} f_{j+1}\right) \times r_{cj} \tag{4}$$

First, we note that, for the last link $j = n$, both f_{j+1} and τ_{j+1} are zero. Thus:

$$f_n = -m_n g^n \tag{5}$$

Then, by substitution, we have:

$$\begin{aligned} f_{n-1} &= -m_{n-1} g^{n-1} + \mathrm{R}^{n-1}_n \left(-m_n g^n\right) & (6)\\ &= -m_{n-1} g^{n-1} - m_n g^{n-1} & (7)\\ &= -\left(m_{n-1} + m_n\right) g^{n-1} & (8) \end{aligned}$$

Thus, we find an explicit expression for the force f_j on each link:

$$f_j = -M_j\, g^j \;\text{ for }\; M_j = \sum_{k=j}^{n} m_k \tag{9}$$

(of course, this seems obvious in retrospect, but it's nice to derive it explicitly as well ...)

Next, we turn to the torque equation.

$$\begin{aligned} \vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} &= -\left(-M_j\, g^j\right) \times (r_{cj} - r_{mj}) + \left(\mathrm{R}^j_{j+1}\left(-M_{j+1}\, g^{j+1}\right)\right) \times r_{cj} & (10)\\ &= g^j \times \left(M_j \left[r_{cj} - r_{mj}\right]\right) - g^j \times \left(M_{j+1}\, r_{cj}\right) & (11)\\ &= g^j \times \left(M_j\, r_{cj} - M_j\, r_{mj} - M_{j+1}\, r_{cj}\right) & (12)\\ &= g^j \times \left(m_j\, r_{cj} - M_j\, r_{mj}\right) & (13) \end{aligned}$$

Thus, we arrive at the following static governing equation:

$$\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} = g^j \times \mu_j \;\text{ for }\; \mu_j = m_j\, r_{cj} - M_j\, r_{mj} \tag{14}$$

3 Calibration - Coupled Matrix

3.1 Denavit-Hartenberg Representation

Next, we take advantage of our Denavit-Hartenberg frame formulation to write each torque:

$$\vec{\tau}_j = \mathrm{R}^j_{j-1} \begin{bmatrix} a_j \\ b_j \\ \tau_j \end{bmatrix} \quad \text{and} \quad [\,\vec{a}\,]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{15}$$

$$\mathrm{R}^j_{j-1} \begin{bmatrix} a_j \\ b_j \\ \tau_j \end{bmatrix} - \begin{bmatrix} a_{j+1} \\ b_{j+1} \\ \tau_{j+1} \end{bmatrix} = \left[g^j\right]_\times \mu_j \tag{16}$$

3.2 Matrix Formulation - Single Link j

Next, we get things in matrix language so we can design a matrix solution. We make a couple of useful definitions to separate the rotation matrix for multiplication with known and unknown quantities:

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad c_j = \begin{bmatrix} a_j \\ b_j \end{bmatrix} \tag{17}$$

$$\mathrm{R}^j_{j-1} L c_j + \mathrm{R}^j_{j-1} Z \tau_j - L c_{j+1} - Z \tau_{j+1} = \left[g^j\right]_\times \begin{bmatrix} \mu_j \end{bmatrix} \tag{18}$$

$$\mathrm{R}^j_{j-1} Z \tau_j - Z \tau_{j+1} = \left[g^j\right]_\times \begin{bmatrix} \mu_j \end{bmatrix} - \mathrm{R}^j_{j-1} L c_j + L c_{j+1} \tag{19}$$

$$\begin{bmatrix} \mathrm{R}^j_{j-1} Z \tau_j - Z \tau_{j+1} \end{bmatrix} = \begin{bmatrix} [g^j]_\times & \begin{bmatrix} -\mathrm{R}^j_{j-1} L & L \end{bmatrix} \end{bmatrix} \begin{bmatrix} \mu_j \\ c_j \\ c_{j+1} \end{bmatrix} \tag{20}$$

We see that this is now in the form $\vec{y} = \mathrm{GT}\ \vec{x}$, with $\vec{x}$ the parameter matrix to be determined by regression. Note that the vector μ is pose-independent, while the torques a and b are different for each pose.

3.3 Matrix Formulation - Coupled Links, Single Pose

We now determine a sample GT matrix for a manipulator with $n = 4$ links, going through 5 poses A, B, C, D, and E.

For pose A, we form the following G matrix:

$$^{A}G = \begin{bmatrix} \left[{}^{A}g^{1}\right]_{\times} & 0 & 0 & 0 \\ 0 & \left[{}^{A}g^{2}\right]_{\times} & 0 & 0 \\ 0 & 0 & \left[{}^{A}g^{3}\right]_{\times} & 0 \\ 0 & 0 & 0 & \left[{}^{A}g^{4}\right]_{\times} \end{bmatrix} \tag{21}$$

For pose A, we also form the following T matrix:

$$^{A}T = \begin{bmatrix} -{}^{A}\mathrm{R}_{0}^{1}L & L & 0 & 0 \\ 0 & -{}^{A}\mathrm{R}_{1}^{2}L & L & 0 \\ 0 & 0 & -{}^{A}\mathrm{R}_{2}^{3}L & L \\ 0 & 0 & 0 & -{}^{A}\mathrm{R}_{3}^{4}L \end{bmatrix} \tag{22}$$

... along with a known torque vector y and unknown torque parameter vector p:

$$^{A}y = \begin{bmatrix} {}^{A}\mathrm{R}_{0}^{1}Z{}^{A}\tau_{1} - Z{}^{A}\tau_{2} \\ {}^{A}\mathrm{R}_{1}^{2}Z{}^{A}\tau_{2} - Z{}^{A}\tau_{3} \\ {}^{A}\mathrm{R}_{2}^{3}Z{}^{A}\tau_{3} - Z{}^{A}\tau_{4} \\ {}^{A}\mathrm{R}_{3}^{4}Z{}^{A}\tau_{4} - 0 \end{bmatrix} \quad \text{for} \quad {}^{A}p = \begin{bmatrix} {}^{A}c_{1} \\ {}^{A}c_{2} \\ {}^{A}c_{3} \\ {}^{A}c_{4} \end{bmatrix} \tag{23}$$

For pose A, then, we have:

$$^{A}y = \begin{bmatrix} {}^{A}G & {}^{A}T \end{bmatrix} \begin{bmatrix} U \\ {}^{A}p \end{bmatrix} \tag{24}$$

3.4 Matrix Formulation - Multiple Poses

However, this parameterization is ambiguous, since the known torque vector y is of length $3n$, while the parameter vector $[U, p]^T$ is of length $3n + 2n = 5n$. This reflects the fact that one pose is insufficient to uniquely determine the parameterization. However, since the parameters μ_j are independent of pose, the solution can be found with a sufficient number of poses k.

For multiple poses, we form the GT matrix as follows:

$$\begin{bmatrix} {}^{A}y \\ {}^{B}y \\ {}^{C}y \\ {}^{D}y \\ {}^{E}y \end{bmatrix} = \begin{bmatrix} {}^{A}G & {}^{A}T & 0 & 0 & 0 & 0 \\ {}^{B}G & 0 & {}^{B}T & 0 & 0 & 0 \\ {}^{C}G & 0 & 0 & {}^{C}T & 0 & 0 \\ {}^{D}G & 0 & 0 & 0 & {}^{D}T & 0 \\ {}^{E}G & 0 & 0 & 0 & 0 & {}^{E}T \end{bmatrix} \begin{bmatrix} U \\ {}^{A}p \\ {}^{B}p \\ {}^{C}p \\ {}^{D}p \\ {}^{E}p \end{bmatrix} \tag{25}$$

Thus, for k poses, the known torque vector y is of length $\mathrm{len}\{y\} = 3nk$, while the parameter vector p is of length $\mathrm{len}\{p\} = 3n + 2nk$. Thus, you need at least $k = 3$ poses for a complete parameterization, and you can use more poses and a regression analysis to determine the best-fit parameters.

4 Calibration - Iterative Algorithm

While the coupled matrix method described above works in theory, it does not yield very precise solutions under linear regression, especially for the end links. There may be a way to weight the regression in some way, but attempts so far have been unsuccessful. Therefore, we take advantage of the directional coupling of the pose matrices G and T and offer a different solution method.

4.1 Matrix Forumulation

We start with the single link, single pose equation (Eqn. 19), and rewrite:

$$\mathrm{R}^j_{j-1} Z\tau_j - Z\tau_{j+1} - Lc_{j+1} = \left[g^j\right]_\times \left[\ \mu_j\ \right] - \mathrm{R}^j_{j-1} Lc_j \tag{26}$$

We then extend this equation across multiple poses, for a single link j:

$$\begin{bmatrix} {}^A\mathrm{R}^j_{j-1} Z^A\tau_j - Z^A\tau_{j+1} \\ {}^B\mathrm{R}^j_{j-1} Z^B\tau_j - Z^B\tau_{j+1} \\ {}^C\mathrm{R}^j_{j-1} Z^C\tau_j - Z^C\tau_{j+1} \\ \vdots \end{bmatrix} - \begin{bmatrix} 0 & L & 0 & 0 & \dots \\ 0 & 0 & L & 0 & \dots \\ 0 & 0 & 0 & L & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \mu_{j+1} \\ {}^Ac_{j+1} \\ {}^Bc_{j+1} \\ {}^Cc_{j+1} \\ \vdots \end{bmatrix} \tag{27}$$

$$= \begin{bmatrix} \left[{}^Ag^j\right]_\times & -{}^A\mathrm{R}^j_{j-1} L & 0 & 0 & \dots \\ \left[{}^Bg^j\right]_\times & 0 & -{}^B\mathrm{R}^j_{j-1} L & 0 & \dots \\ \left[{}^Cg^j\right]_\times & 0 & 0 & -{}^C\mathrm{R}^j_{j-1} L & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \mu_j \\ {}^Ac_j \\ {}^Bc_j \\ {}^Cc_j \\ \vdots \end{bmatrix} \tag{28}$$

5 Second Moment Calculation

Note: This is only in here until it gets its own document.

5.1 Dynamics

Assumptions: No external forces nor torques. Start with the Newton-Euler equations from above:

$$f_j - \mathrm{R}^j_{j+1} f_{j+1} + m_j g^j = m_j a^j_{cj} \tag{29}$$

$$\vec{\tau}_j - \mathrm{R}^j_{j+1} \vec{\tau}_{j+1} + f_j \times (r_{cj} - r_{mj}) - \left(\mathrm{R}^j_{j+1} f_{j+1}\right) \times r_{cj} = I_j \alpha_j + \omega_j \times (I_j \omega_j) \tag{30}$$

First, we derive a closed-form solution to the force equation:

$$f_n = -m_n g^n + m_n a^n_{cn} \tag{31}$$

$$\begin{aligned} f_{n-1} &= \mathrm{R}^{n-1}_n \left(-m_n g^n + m_n a^n_{cn}\right) - m_{n-1} g^{n-1} + m_{n-1} a^{n-1}_{cn-1} && (32) \\ &= -m_n g^{n-1} + m_n a^{n-1}_{cn} - m_{n-1} g^{n-1} + m_{n-1} a^{n-1}_{cn-1} && (33) \\ &= -M_{n-1} g^{n-1} + m_{n-1} a^{n-1}_{cn-1} + m_n a^{n-1}_{cn} && (34) \end{aligned}$$

Thus, for a general link j, we have:

$$f_j = -M_j g^j + \sum_{h=j}^{n} m_h a^j_{ch} \tag{35}$$

Next, substitute into the torque equation:

$$\begin{aligned} \vec{\tau}_j - \mathrm{R}^j_{j+1} \vec{\tau}_{j+1} &+ \left(-M_j g^j + \sum_{h=j}^{n} m_h a^j_{ch}\right) \times (r_{cj} - r_{mj}) && (36) \\ &- \mathrm{R}^j_{j+1} \left(-M_{j+1} g^{j+1} + \sum_{h=j+1}^{n} m_h a^{j+1}_{ch}\right) \times r_{cj} && (37) \\ &= I_j \alpha_j + \omega_j \times (I_j \omega_j) && (38) \end{aligned}$$

Carry though the f_{j+1} rotation matrix, collect the gravity terms, and split the f_j cross-product:

$$\begin{aligned} \vec{\tau}_j - \mathrm{R}^j_{j+1} \vec{\tau}_{j+1} &+ g^j \times (-M_j r_{cj} + M_j r_{mj}) + g^j \times (M_{j+1} r_{cj}) && (39) \\ &+ \sum_{h=j}^{n} m_h a^j_{ch} \times r_{cj} - \sum_{h=j}^{n} m_h a^j_{ch} \times r_{mj} && (40) \\ &- \sum_{h=j+1}^{n} m_h a^j_{ch} \times r_{cj} && (41) \\ &= I_j \alpha_j + \omega_j \times (I_j \omega_j) && (42) \end{aligned}$$

Collect the gravity terms, and perform the r_{cj} subtraction:

$$\vec{\tau}_j - \mathrm{R}^j_{j+1} \vec{\tau}_{j+1} + \mu_j \times g^j + m_j a^j_{cj} \times r_{cj} - \sum_{h=j}^{n} m_h a^j_{ch} \times r_{mj} = I_j \alpha_j + \omega_j \times (I_j \omega_j) \tag{43}$$

5.2 Single-Joint Motion - Single Acceleration

Next, let's examine the relationship between the acceleration vectors a^j_{cj}, a^j_{cj+1}, etc. These notes are taken loosely from Spong pp. 274 - 279, although various typos warranted rederivation.

Assume that only one joint k is moving, while all other joints are held still. Thus, only links $j = k, k+1, \dots n$ are accelerating, along with their corresponding frames.

First, we note that the frame velocities v and the COM velocities v_c of links $j < k$ in the inertial frame $j = 0$ are zero:

$$v^0_{kj} = v^0_{ckj} = 0 \text{ for } j < k \tag{44}$$

Next, we examine the inertial velocities for links $j \geq k$. Note, first, that the rotational velocities ω_j for $j \geq k$ expressed in the inertial frame are equal, and are simply termed ω^0 below. Luckily, velocities just add (since $v << c$).

$$\begin{aligned}
v^0_{ckk} &= \omega^0 \times \mathrm{R}^0_k \left(r_{ck} - r_{mk}\right) && (45)\\
v^0_{kk} &= \omega^0 \times \mathrm{R}^0_k \left(-r_{mk}\right) && (46)\\
v^0_{ckk+1} &= \omega^0 \times \mathrm{R}^0_k \left(-r_{mk}\right) + \omega^0 \times \mathrm{R}^0_{k+1} \left(r_{ck+1} - r_{mk+1}\right) && (47)\\
v^0_{kk+1} &= \omega^0 \times \mathrm{R}^0_k \left(-r_{mk}\right) + \omega^0 \times \mathrm{R}^0_{k+1} \left(-r_{mk+1}\right) && (48)\\
&\vdots && (49)\\
v^0_{ckj} &= \omega^0 \times \left[\mathrm{R}^0_j r_{cj} - \sum_{i=k}^{j} \mathrm{R}^0_i r_{mi}\right] \text{ for } j \geq k && (50)
\end{aligned}$$

Again, this seems obvious in retrospect, but it's nice to derive it. Before we start differentiating, we split each rotation matrix into time-dependent and time-independent parts. Since only joint k is moving, R^{k-1}_k is the only time-dependent rotation matrix.

$$\begin{aligned}
v^0_{ckj} &= \omega^0 \times \left[\mathrm{R}^0_{k-1}\mathrm{R}^{k-1}_k\mathrm{R}^k_j r_{cj} - \sum_{i=k}^{j} \mathrm{R}^0_{k-1}\mathrm{R}^{k-1}_k\mathrm{R}^k_i r_{mi}\right] && (51)\\
&= S(\omega^0)\mathrm{R}^0_{k-1}\mathrm{R}^{k-1}_k \left[\mathrm{R}^k_j r_{cj} - \sum_{i=k}^{j} \mathrm{R}^k_i r_{mi}\right] \text{ for } j \geq k && (52)
\end{aligned}$$

Next, we take the time derivative of the inertial velocities to get the inertial accelerations, and, by definition of angular velocity, we can make the substitution $\dot{\mathrm{R}} = S(\omega^0)\mathrm{R}$:

$$a^0_{ckj} = \left[S(\dot{\omega}^0)\mathrm{R}^0_{k-1}\mathrm{R}^{k-1}_k + S(\omega^0)\mathrm{R}^0_{k-1}\dot{\mathrm{R}}^{k-1}_k\right]\left[\mathrm{R}^k_j r_{cj} - \sum_{i=k}^{j} \mathrm{R}^k_i r_{mi}\right] \text{ for } j \geq k \tag{53}$$

$$a^p_{ckj} = \mathrm{R}^p_0 \left[S(\dot{\omega}^0)\mathrm{R}^0_k + S(\omega^0)\mathrm{R}^0_{k-1}S(\omega^0)\mathrm{R}^{k-1}_k\right]\left[\mathrm{R}^k_j r_{cj} - \sum_{i=k}^{j} \mathrm{R}^k_i r_{mi}\right] \text{ for } j \geq k \tag{54}$$

$$a^p_{ckj} = W^p_k(\omega^0, \dot{\omega}^0)\left[\mathrm{R}^k_j r_{cj} - \sum_{i=k}^{j} \mathrm{R}^k_i r_{mi}\right] \text{ for } j \geq k \tag{55}$$

$$\text{with: } W^p_k(\omega^0, \dot{\omega}^0) = \mathrm{R}^p_0 \left[S(\dot{\omega}^0)\mathrm{R}^0_k + S(\omega^0)\mathrm{R}^0_{k-1}S(\omega^0)\mathrm{R}^{k-1}_k\right] \tag{56}$$

5.3 Single-Joint Motion - Acceleration Sums

Now that we have an expression for the acceleration a_c of any link in a single-joint motion system, we turn to deriving an expression for the sums of mass-accelerations present in the general torque equation:

$$\sum_{h=j}^{n} m_h a_{ckh}^p \tag{57}$$

Since only links past the active joint k are accelerating, we assume $j \geq k$, and use the acceleration value derived earlier for each acceleration:

$$\sum_{h=j}^{n} m_h W_k^p(\omega^0, \dot{\omega}^0) \left[\mathrm{R}_h^k r_{ch} - \sum_{i=k}^{h} \mathrm{R}_i^k r_{mi} \right] \tag{58}$$

We see immediately that the matrix W_k^p is independent of h:

$$W_k^0(\omega^0, \dot{\omega}^0) \sum_{h=j}^{n} m_h \left[\mathrm{R}_h^k r_{ch} - \sum_{i=k}^{h} \mathrm{R}_i^k r_{mi} \right] \tag{59}$$

Next, we rearrange the sums:

$$W_k^p \left[\sum_{h=j}^{n} \mathrm{R}_h^k m_h r_{ch} - \sum_{h=j}^{n} \sum_{i=k}^{h} \mathrm{R}_i^k m_h r_{mi} \right] \tag{60}$$

$$W_k^p \left[\sum_{h=j}^{n} \mathrm{R}_h^k m_h r_{ch} - \sum_{h=j}^{n} \left[\sum_{i=j}^{h} \mathrm{R}_i^k m_h r_{mi} + \sum_{i=k}^{j-1} \mathrm{R}_i^k m_h r_{mi} \right] \right] \tag{61}$$

$$W_k^p \left[\sum_{h=j}^{n} \mathrm{R}_h^k m_h r_{ch} - \sum_{h=j}^{n} \sum_{i=j}^{h} \mathrm{R}_i^k m_h r_{mi} - \sum_{h=j}^{n} \sum_{i=k}^{j-1} \mathrm{R}_i^k m_h r_{mi} \right] \tag{62}$$

Next, a quick change-of-base ...

$$W_k^p \left[\sum_{h=j}^{n} \mathrm{R}_h^k m_h r_{ch} - \sum_{i=j}^{n} \sum_{h=i}^{n} \mathrm{R}_i^k m_h r_{mi} - \sum_{i=k}^{j-1} \sum_{h=j}^{n} \mathrm{R}_i^k m_h r_{mi} \right] \tag{63}$$

$$W_k^p \left[\sum_{h=j}^{n} \mathrm{R}_h^k m_h r_{ch} - \sum_{i=j}^{n} \mathrm{R}_i^k r_{mi} \sum_{h=i}^{n} m_h - \sum_{i=k}^{j-1} \mathrm{R}_i^k r_{mi} \sum_{h=j}^{n} m_h \right] \tag{64}$$

$$W_k^p \left[\sum_{i=j}^{n} \mathrm{R}_i^k m_i r_{ci} - \sum_{i=j}^{n} \mathrm{R}_i^k M_i r_{mi} - \sum_{i=k}^{j-1} \mathrm{R}_i^k M_j r_{mi} \right] \tag{65}$$

$$W_k^p \left[\sum_{i=j}^{n} \mathrm{R}_i^k \left(m_i r_{ci} - M_i r_{mi} \right) - \sum_{i=k}^{j-1} \mathrm{R}_i^k M_j r_{mi} \right] \tag{66}$$

Therefore, we have:

$$\sum_{h=j}^{n} m_h a_{ckh}^p = W_k^p \left[\sum_{i=j}^{n} \mathrm{R}_i^k \mu_i - \sum_{i=k}^{j-1} \mathrm{R}_i^k M_j r_{mi} \right] \tag{67}$$

5.4 Inertial Link Solution

Turning back to the general torque equation, we make the assumption of a single-joint motion system, and begin by examining the case where $j < k$ (i.e. link j is not moving).

$$\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j + m_j a^j_{cj} \times r_{cj} - \sum_{h=j}^{n} m_h a^j_{ch} \times r_{mj} = I_j \alpha_j + \omega_j \times (I_j \omega_j) \tag{68}$$

Since, for single-joint motion at joint $j = k$, links $j < k$ are not moving, we take $a_{cj} = 0$ for $j < k$. Thus:

$$\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j - \sum_{h=k}^{n} m_h a^j_{ckh} \times r_{mj} = I_j \alpha_j + \omega_j \times (I_j \omega_j) \tag{69}$$

By making the substitution from earlier for the mass-acceleration sum, we have:

$$\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j - W^j_k \left[\sum_{i=k}^{n} \mathrm{R}^k_i \mu_i - \sum_{i=k}^{k-1} \mathrm{R}^k_i M_j r_{mi} \right] \times r_{mj} = I_j \alpha_j + \omega_j \times (I_j \omega_j) \tag{70}$$

We see that the second sum does not exist, and we are left with:

$$\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j - W^j_k \sum_{i=k}^{n} \mathrm{R}^k_i \mu_i \times r_{mj} = I_j \alpha_j + \omega_j \times (I_j \omega_j) \tag{71}$$

Now, in a particular manipulator pose ${}^X P$ of links above the active joint, we see that the sum quantity is constant. So, we define ${}^X\eta_{kj}$ as:

$${}^X\eta_{kj} = -\sum_{i=k}^{n} {}^X\mathrm{R}^k_i \mu_i \times r_{mj} \ \text{ for } \ j < k \tag{72}$$

Then, we have:

$$\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j + W^j_k {}^X\eta_{jk} = I_j \alpha_j + \omega_j \times (I_j \omega_j) \tag{73}$$

5.5 Moving Link Solution

The moving case is a bit more complex, but we start with the same general torque balance equation. In this case, $j \geq k$, so all acceleration terms are nonzero.

$$\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j + m_j a^j_{ckj} \times r_{cj} - \sum_{h=j}^{n} m_h a^j_{ckh} \times r_{mj} = I_j\alpha_j + \omega_j \times (I_j\omega_j) \tag{74}$$

We again use the mass-acceleration sum substitution:

$$\begin{aligned}
\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j &+ m_j W^j_k \left[\mathrm{R}^k_j r_{cj} - \sum_{i=k}^{j} \mathrm{R}^k_i r_{mi}\right] \times r_{cj} &(75)\\
&- W^j_k \left[\sum_{i=j}^{n} \mathrm{R}^k_i \mu_i - \sum_{i=k}^{j-1} \mathrm{R}^k_i M_j r_{mi}\right] \times r_{mj} &(76)\\
&= I_j\alpha_j + \omega_j \times (I_j\omega_j) &(77)
\end{aligned}$$

Next, we distribute across the cross products:

$$\begin{aligned}
\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j &+ W^j_k m_j \mathrm{R}^k_j r_{cj} \times r_{cj} &(78)\\
&- W^j_k m_j \sum_{i=k}^{j} \mathrm{R}^k_i r_{mi} \times r_{cj} &(79)\\
&- W^j_k \sum_{i=j}^{n} \mathrm{R}^k_i \mu_i \times r_{mj} &(80)\\
&+ W^j_k M_j \sum_{i=k}^{j-1} \mathrm{R}^k_i r_{mi} \times r_{mj} &(81)\\
&= I_j\alpha_j + \omega_j \times (I_j\omega_j) &(82)
\end{aligned}$$

We note that $a \times a = 0$ and $a \times b = -b \times a$, further regroup terms, and introduce an extra term to synchronize the sum indices:

$$\begin{aligned}
\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j &- W^j_k \sum_{i=j}^{n} \mathrm{R}^k_i \mu_i \times r_{mj} &(83)\\
&+ W^j_k m_j \sum_{i=k}^{j} \left(r_{cj} \times \mathrm{R}^k_i r_{mi}\right) &(84)\\
&- W^j_k M_j \sum_{i=k}^{j-1} \left(r_{mj} \times \mathrm{R}^k_i r_{mi}\right) &(85)\\
&- W^j_k M_j \left(r_{mj} \times \mathrm{R}^k_j r_{mj}\right) &(86)\\
&+ W^j_k M_j \left(r_{mj} \times \mathrm{R}^k_j r_{mj}\right) &(87)\\
&= I_j\alpha_j + \omega_j \times (I_j\omega_j) &(88)
\end{aligned}$$

We can now combine the sums:

$$
\begin{aligned}
\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j \quad &- \quad W^j_k \sum_{i=j}^{n} \mathrm{R}^k_i \mu_i \times r_{mj} &(89)\\
&+ \quad W^j_k m_j \sum_{i=k}^{j} \left(r_{cj} \times \mathrm{R}^k_i r_{mi} \right) &(90)\\
&- \quad W^j_k M_j \sum_{i=k}^{j} \left(r_{mj} \times \mathrm{R}^k_i r_{mi} \right) &(91)\\
&+ \quad W^j_k M_j \left(r_{mj} \times \mathrm{R}^k_j r_{mj} \right) = I_j \alpha_j + \omega_j \times (I_j \omega_j) &(92)
\end{aligned}
$$

$$
\begin{aligned}
\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j \quad &- \quad W^j_k \sum_{i=j}^{n} \mathrm{R}^k_i \mu_i \times r_{mj} &(93)\\
&+ \quad W^j_k \sum_{i=k}^{j} \left(m_j r_{cj} \times \mathrm{R}^k_i r_{mi} - M_j r_{mj} \times \mathrm{R}^k_i r_{mi} \right) &(94)\\
&+ \quad W^j_k M_j \left(r_{mj} \times \mathrm{R}^k_j r_{mj} \right) = I_j \alpha_j + \omega_j \times (I_j \omega_j) &(95)
\end{aligned}
$$

We further group the inside of the sum, and find another μ:

$$
\begin{aligned}
\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j \quad &- \quad W^j_k \sum_{i=j}^{n} \mathrm{R}^k_i \mu_i \times r_{mj} &(96)\\
&+ \quad W^j_k \sum_{i=k}^{j} \left(\mu_j \times \mathrm{R}^k_i r_{mi} \right) &(97)\\
&+ \quad W^j_k M_j \left(r_{mj} \times \mathrm{R}^k_j r_{mj} \right) = I_j \alpha_j + \omega_j \times (I_j \omega_j) &(98)
\end{aligned}
$$

We then pull out the velocity- and acceleration-dependent W^j_k:

$$
\begin{aligned}
& \vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j &(99)\\
+ \quad & W^j_k \left[\mu_j \times \sum_{i=k}^{j} \mathrm{R}^k_i r_{mi} - \sum_{i=j}^{n} \mathrm{R}^k_i \mu_i \times r_{mj} + M_j \left(r_{mj} \times \mathrm{R}^k_j r_{mj} \right) \right] &(100)\\
= \quad & I_j \alpha_j + \omega_j \times (I_j \omega_j) &(101)
\end{aligned}
$$

We note that the bracketed quantity is independent of the motion of the manipulator for a fixed pose ${}^X P$ above the single moving joint k, so we define ${}^X\eta_{kj}$ as follows:

$$
{}^X\eta_{kj} = \mu_j \times \sum_{i=k}^{j} {}^X\mathrm{R}^k_i r_{mi} - \sum_{i=j}^{n} {}^X\mathrm{R}^k_i \mu_i \times r_{mj} + M_j \left(r_{mj} \times {}^X\mathrm{R}^k_j r_{mj} \right) \quad \text{for } \; j \geq k \tag{102}
$$

Thus, the torque equation for a moving link reduces to:

$$
\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j + W^j_k {}^X\eta_{kj} = I_j \alpha_j + \omega_j \times (I_j \omega_j) \tag{103}
$$

We see that this is equivalent to the inertial-link case, simply with a more expanded definition of the vector ${}^X\eta_{kj}$.

5.6 Torque Equation Summary

Thus, we have the single-joint motion torque equation:

$$\vec{\tau}_j - \mathrm{R}_{j+1}^{j}\vec{\tau}_{j+1} + \mu_j \times g^j + W_k^j(\omega^0, \dot{\omega}^0)^X \eta_{kj} = I_j\alpha_j + \omega_j \times (I_j\omega_j) \tag{104}$$

The definition of the vector ${}^X\eta_{kj}$ for pose X depends on whether the link j is moving.

$${}^X\eta_{kj} = \begin{cases} -\sum_{i=k}^{n} {}^X\mathrm{R}_i^k \mu_i \times r_{mj} & \text{for } j < k \\ \mu_j \times \sum_{i=k}^{j} {}^X\mathrm{R}_i^k r_{mi} - \sum_{i=j}^{n} {}^X\mathrm{R}_i^k \mu_i \times r_{mj} + M_j \left(r_{mj} \times {}^X\mathrm{R}_j^k r_{mj} \right) & \text{for } j \geq k \end{cases} \tag{105}$$

5.7 The Inertia Matrix

Next, we turn our attention to the inertia matrix I:

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \text{ with } \begin{matrix} I_{xy} = I_{yx} \\ I_{xz} = I_{zx} \\ I_{yz} = I_{zy} \end{matrix} \tag{106}$$

We examine the right-hand side of the torque equation:

$$I\alpha + \omega \times (I\omega) = I\alpha + S(\omega)I\omega \tag{107}$$

We carry out the matrix multiplication by hand:

$$\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} + \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{108}$$

$$\begin{bmatrix} I_{xx}\alpha_x + I_{xy}\alpha_y + I_{xz}\alpha_z \\ I_{xy}\alpha_x + I_{yy}\alpha_y + I_{yz}\alpha_z \\ I_{xz}\alpha_x + I_{yz}\alpha_y + I_{zz}\alpha_z \end{bmatrix} + \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z \end{bmatrix} \tag{109}$$

$$\begin{bmatrix} I_{xx}\alpha_x + I_{xy}\alpha_y + I_{xz}\alpha_z \\ I_{xy}\alpha_x + I_{yy}\alpha_y + I_{yz}\alpha_z \\ I_{xz}\alpha_x + I_{yz}\alpha_y + I_{zz}\alpha_z \end{bmatrix} + \begin{bmatrix} -I_{xy}\omega_x\omega_z - I_{yy}\omega_y\omega_z - I_{yz}\omega_z^2 + I_{xz}\omega_x\omega_y + I_{yz}\omega_y^2 + I_{zz}\omega_y\omega_z \\ I_{xx}\omega_x\omega_z + I_{xy}\omega_y\omega_z + I_{xz}\omega_z^2 - I_{xz}\omega_x^2 - I_{yz}\omega_x\omega_y - I_{zz}\omega_x\omega_z \\ -I_{xx}\omega_x\omega_y - I_{xy}\omega_y^2 - I_{xz}\omega_y\omega_z + I_{xy}\omega_x^2 + I_{yy}\omega_x\omega_y + I_{yz}\omega_x\omega_z \end{bmatrix} \tag{110}$$

$$\begin{bmatrix} \alpha_x & -\omega_y\omega_z & \omega_y\omega_z & \alpha_y - \omega_x\omega_z & \alpha_z + \omega_x\omega_y & \omega_y^2 - \omega_z^2 \\ \omega_x\omega_z & \alpha_y & -\omega_x\omega_z & \alpha_x + \omega_y\omega_z & \omega_z^2 - \omega_x^2 & \alpha_z - \omega_x\omega_y \\ -\omega_x\omega_y & \omega_x\omega_y & \alpha_z & \omega_x^2 - \omega_y^2 & \alpha_x - \omega_y\omega_z & \alpha_y + \omega_x\omega_z \end{bmatrix} \begin{bmatrix} I_{xx} \\ I_{yy} \\ I_{zz} \\ I_{xy} \\ I_{xz} \\ I_{yz} \end{bmatrix} \tag{111}$$

Thus, we have:

$$I\alpha + \omega \times (I\omega) = A_j(\alpha_j, \omega_j)\vec{I}_j \tag{112}$$

5.8 Matrix Formulation

Next, we start with a matrix formulation, using our old Danavit-Hartenberg equations for joint torques. Of course,

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } c_j = \begin{bmatrix} a_j \\ b_j \end{bmatrix} \tag{113}$$

Note that the vectors ${}^X\eta_{kj}$ are constant during a particular ppse ${}^XP$ of upper links $j = k, k+1 \ldots, n$ as joint k moves. For a particular pose ${}^XP$ for links after the moving joint k, we have the following familiar torque equation, with the inertia matrix substitution derived earlier:

$$\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j + W^j_k(\omega^0, \dot{\omega}^0)^X\eta_{kj} = A_j(\alpha_j, \omega_j)\vec{I}_j \tag{114}$$

In matrix language, this corresponds to:

$$\mathrm{R}^j_{j-1}Lc_j + \mathrm{R}^j_{j-1}Z\tau_j - Lc_{j+1} - Z\tau_{j+1} + S(\mu_j)g^j + W^j_k(\omega^0, \dot{\omega}^0)^X\eta_{kj} = A_j(\alpha_j, \omega_j)\vec{I}_j \tag{115}$$

Regroup, knowns on the left, unknowns on the right:

$$\mathrm{R}^j_{j-1}Z\tau_j - Lc_{j+1} - Z\tau_{j+1} + S(\mu_j)g^j = A_j(\alpha_j, \omega_j)\vec{I}_j - W^j_k(\omega^0, \dot{\omega}^0)^X\eta_{kj} - \mathrm{R}^j_{j-1}Lc^X_j \tag{116}$$

Now, we hold an upper pose ${}^XP$, and perform oscillations about joint k in a number of lower poses ${}^X_AP$, ${}^X_BP$, ${}^X_CP$ etc. collecting a number of joint-torque and joint-position data points for each lower pose ${}^X_AP$, ${}^X_BP$, ${}^X_CP$ etc. Thus, we have:

$$\begin{bmatrix} {}^X_Ay_j \\ {}^X_By_j \\ \vdots \\ {}^X_Ny_j \end{bmatrix} = \begin{bmatrix} {}^X_AA_j & -{}^X_AW^j_k & -{}^X_A\mathrm{R}^j_{j-1}L & 0 & 0 & 0 \\ {}^X_BA_j & -{}^X_BW^j_k & 0 & -{}^X_B\mathrm{R}^j_{j-1}L & 0 & 0 \\ \vdots & \vdots & 0 & 0 & \ddots & 0 \\ {}^X_NA_j & -{}^X_NW^j_k & 0 & 0 & 0 & -{}^X_N\mathrm{R}^j_{j-1}L \end{bmatrix} \begin{bmatrix} \vec{I}_j \\ {}^X\eta_{jk} \\ {}^X_Ac_j \\ {}^X_Bc_j \\ \vdots \\ {}^X_Nc_j \end{bmatrix} \tag{117}$$

$$\text{with } {}^X_Ay_j = \mathrm{R}^j_{j-1}Z\tau_j - Lc_{j+1} - Z\tau_{j+1} + S(\mu_j)g^j \tag{118}$$

Or, better yet, to determine μ_j at the same time:

$$\mathrm{R}^j_{j-1}Z\tau_j - Z\tau_{j+1} - Lc_{j+1} = S(g^j)\mu_j + A_j(\alpha_j, \omega_j)\vec{I}_j - W^j_k(\omega^0, \dot{\omega}^0)^X\eta_{kj} - \mathrm{R}^j_{j-1}Lc^X_j \tag{119}$$

$$\begin{bmatrix} {}^X_Ay_j \\ {}^X_By_j \\ \vdots \\ {}^X_Ny_j \end{bmatrix} = \begin{bmatrix} S(g^j) & {}^X_AA_j & -{}^X_AW^j_k & -{}^X_A\mathrm{R}^j_{j-1}L & 0 & 0 & 0 \\ S(g^j) & {}^X_BA_j & -{}^X_BW^j_k & 0 & -{}^X_B\mathrm{R}^j_{j-1}L & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & 0 & \ddots & 0 \\ S(g^j) & {}^X_NA_j & -{}^X_NW^j_k & 0 & 0 & 0 & -{}^X_N\mathrm{R}^j_{j-1}L \end{bmatrix} \begin{bmatrix} \mu_j \\ \vec{I}_j \\ {}^X\eta_{jk} \\ {}^X_Ac_j \\ {}^X_Bc_j \\ \vdots \\ {}^X_Nc_j \end{bmatrix} \tag{120}$$

5.9 For Future Reference (Solving ηs)

Now, let's further look at the torque equation for $j > k$, i.e. compute the torques about the moving join $j = k$.

$$\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j + \mathrm{R}^j_0 W^0_k \left[\mu_j \times \sum_{i=k}^{j} \mathrm{R}^k_i r_{mi} - \sum_{i=j}^{n} \mathrm{R}^k_i \mu_i \times r_{mj}\right] = A_j \vec{I}_j \tag{121}$$

$$\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j + \mathrm{R}^j_0 W^0_k \left[\mu_j \times \sum_{i=k}^{j-1} \mathrm{R}^k_i r_{mi} + \mu_j \times r_{mj} - \sum_{i=j}^{n} \mathrm{R}^k_i \mu_i \times r_{mj}\right] = A_j \vec{I}_j \tag{122}$$

$$\vec{\tau}_j - \mathrm{R}^j_{j+1}\vec{\tau}_{j+1} + \mu_j \times g^j + \mathrm{R}^j_0 W^0_k \left[\mu_j \times \sum_{i=k}^{j-1} \mathrm{R}^k_i r_{mi} + \left(\mu_j - \sum_{i=j}^{n} \mathrm{R}^k_i \mu_i\right) \times r_{mj}\right] = A_j \vec{I}_j \tag{123}$$